Final Review

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https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Exam code

- Exam on Dec 21 (7:30-)8:00-9:40 at Dong Shang Yuan 407 (lecture classroom)
- Finish the exam paper by yourself
- Allowed:
  - Calculator, watch (not smart)
- Not allowed:
  - Books, materials, cheat sheet, …
  - Phones, any smart device
- No entering after 8:30
- Early submission period: 8:30--

Grading policy

- Attendance and participance: 5%
- Assignments: 35%
- Midterm exam: 20%
- Project: 10%
- Final exam: 30%

Covered topics

- Basics
  - Graphs, paths/walks/cycles, bipartite graphs
- Connectivity
- Trees
- Matchings
- Coloring
- Planarity
- Ramsey Theory

Basic Concepts

Graphs

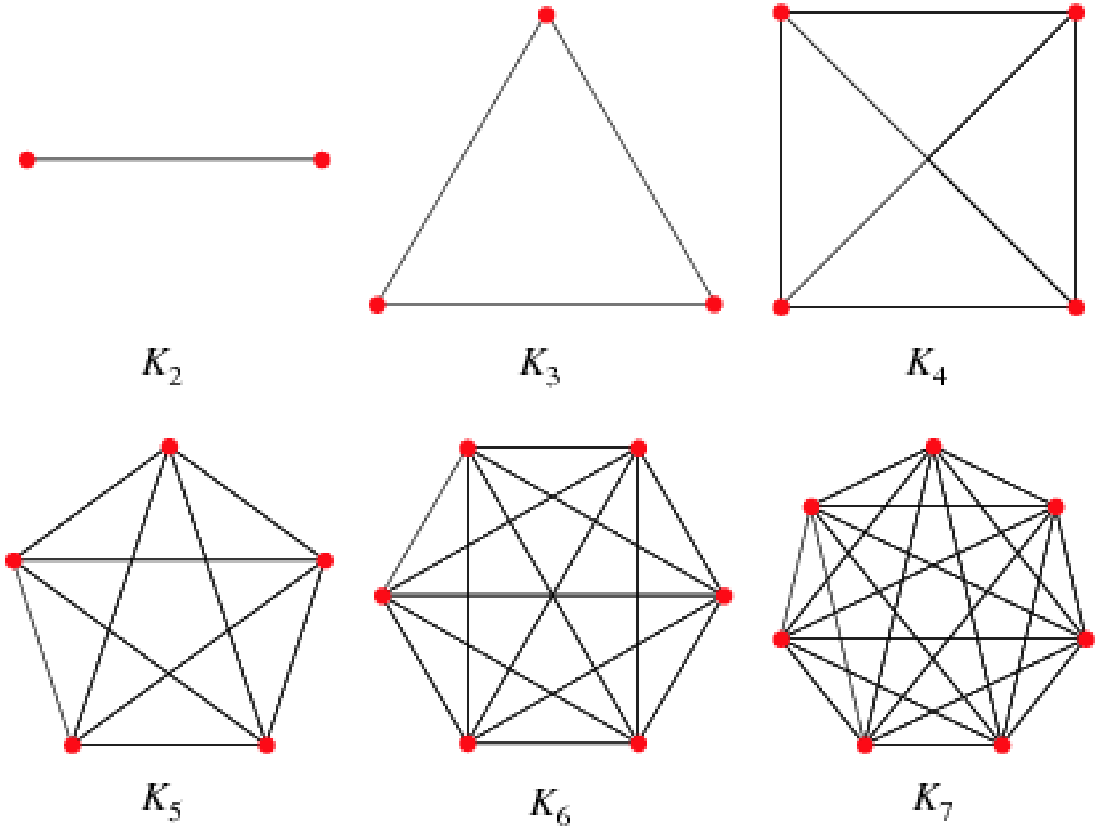
We mainly focus on
Simple graph:
No loops, no multi-edges

- A graph G is a pair (V, E)
  - V: set of vertices
  - E: set of edges
  - $e \in E$ corresponds to a pair of endpoints $x, y \in V$
- Two graphs $G_1 = (V_1, E_1), G_1 = (V_2, E_2)$ are isomorphic if there is a bijection $f: V_1 \to V_2$ s.t.

$$e = \{a, b\} \in E_1 \Leftrightarrow f(e) := \{f(a), f(b)\} \in E_2$$

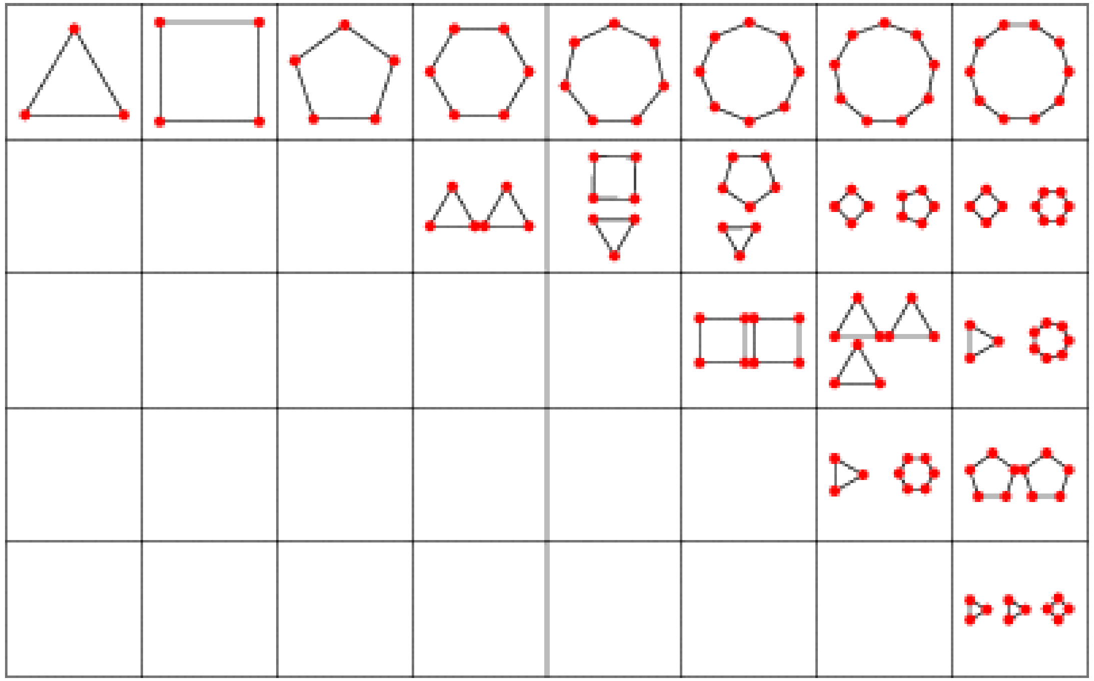
Example: Complete graphs

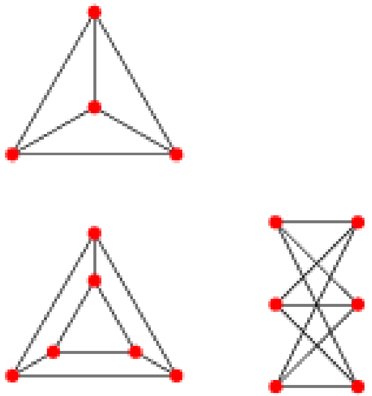
- There is an edge between every pair of vertices

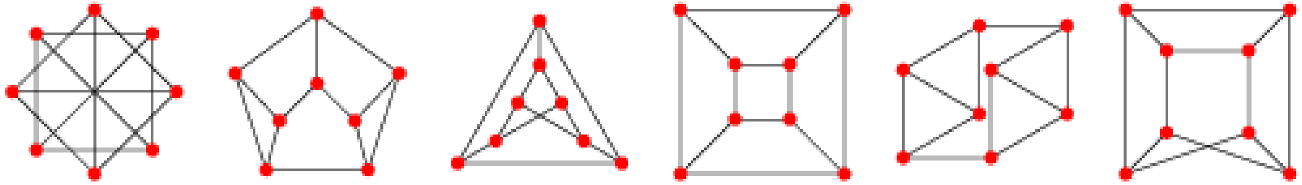


Example: Regular graphs

- Every vertex has the same degree

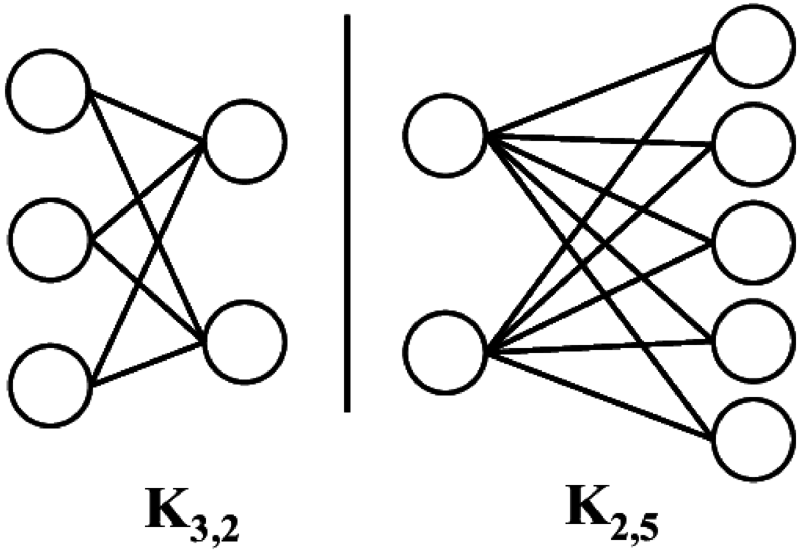






Example: Bipartite graphs

- The vertex set can be partitioned into two sets X and Y such that every edge in G has one end vertex in X and the other in Y
- Complete bipartite graphs



Example (1A, L): Peterson graph

- Show that the following two graphs are same/isomorphic

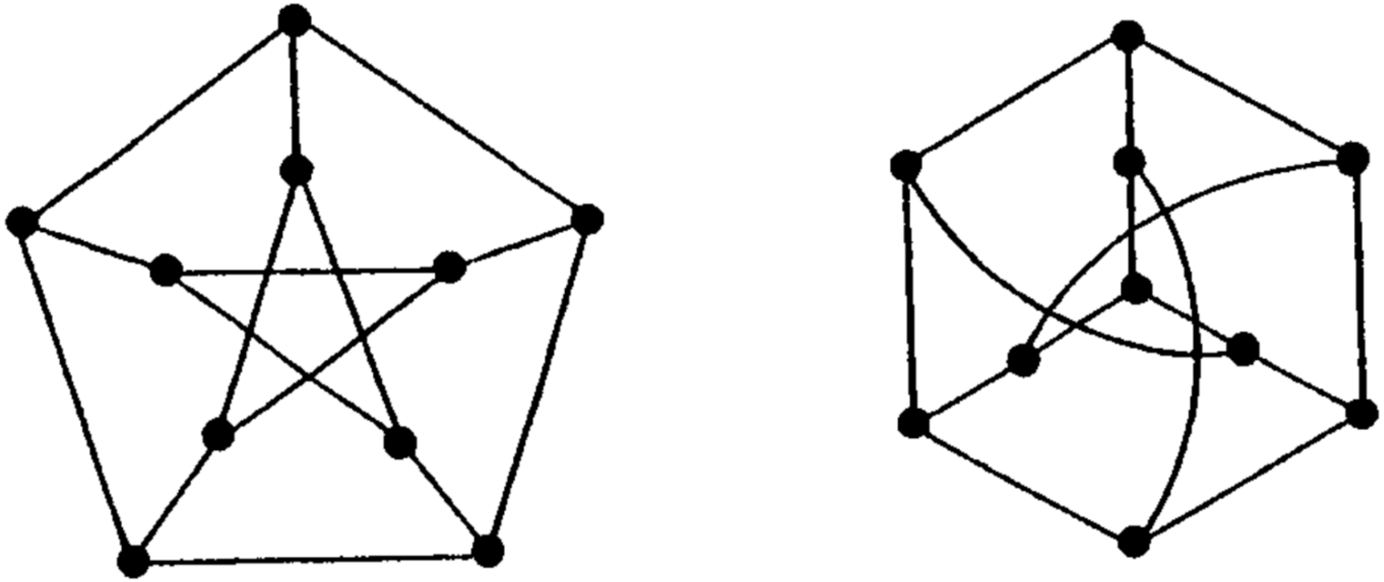
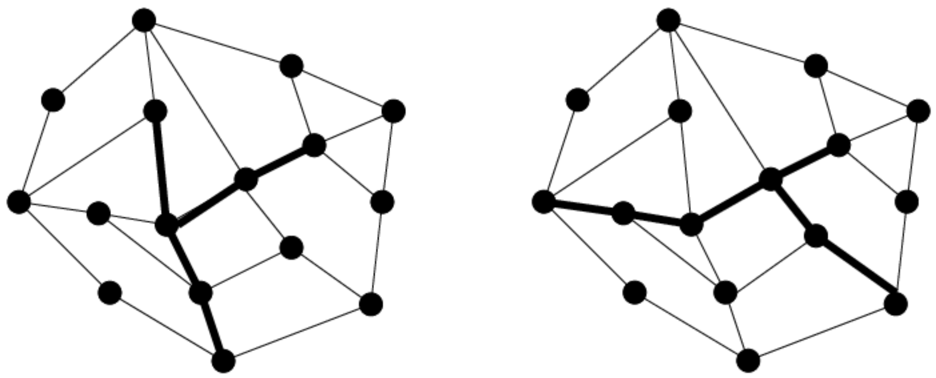
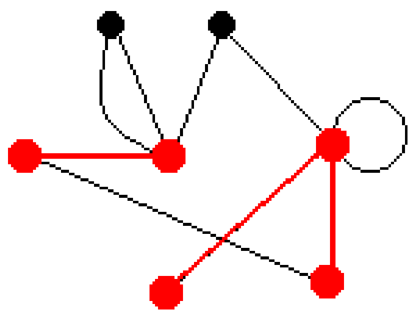


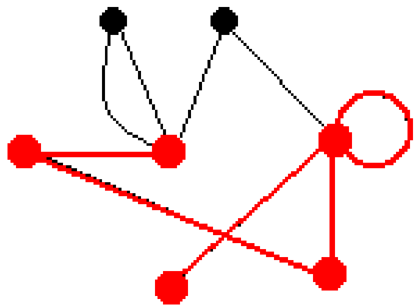
Figure 1.4

Subgraphs

- A subgraph of a graph G is a graph H such that
$$V(H) \subseteq V(G), E(H) \subseteq E(G)$$
and the ends of an edge $e \in E(H)$ are the same as its ends in G
  - H is a spanning subgraph when $V(H) = V(G)$
  - The subgraph of G induced by a subset $S \subseteq V(G)$ is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S

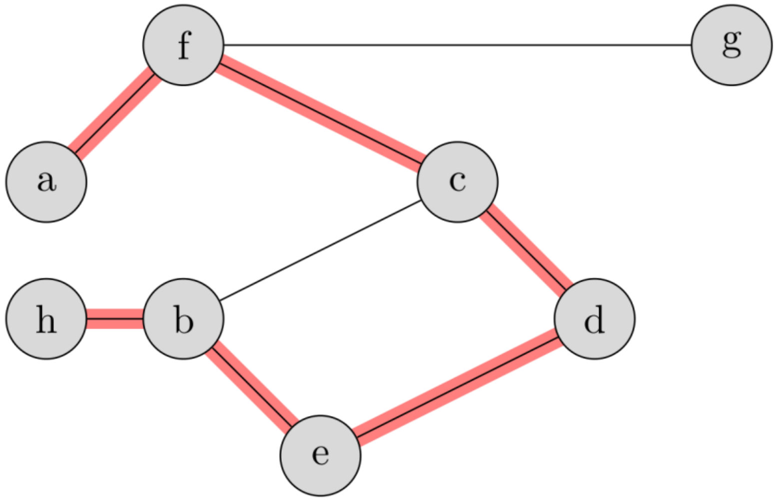

(a) (b)


Subgraph (in red)


Induced Subgraph

Paths (路径)

- A path is a nonempty graph $P = (V, E)$ of the form
$$V = \{x_0, x_1, \dots, x_k\} \quad E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$$
where the x_i are all distinct
- P^k: path of length k (the number of edges)

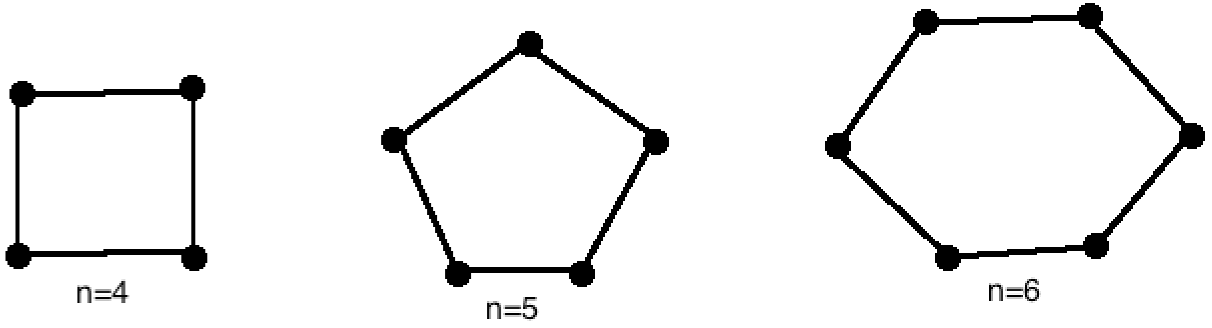


Walk (游走)

- A walk is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \ldots e_k v_k$
  - The vertices not necessarily distinct
  - The length = the number of edges
- Proposition (1.2.5, W) Every u-v walk contains a u-v path

Cycles (环)

- If $P = x_0 x_1 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1}x_0$ is called a cycle
- C^k: cycle of length k (the number of edges/vertices)



- Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Neighbors and degree

- Two vertices $a \neq b$ are called adjacent if they are joined by an edge
  - $N(x)$: set of all vertices adjacent to x
    - neighbors of x
  - A vertex is isolated vertex if it has no neighbors

Handshaking Theorem (Euler 1736)

- Theorem A finite graph G has an even number of vertices with odd degree.
- Proof The degree of x is the number of times it appears in the right column. Thus

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

Degree

- Minimal degree of G: $\delta(G) = \min\{d(v): v \in V\}$
- Maximal degree of G: $\Delta(G) = \min\{d(v): v \in V\}$
- Average degree of G: $d(G) = \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{2|E|}{|V|}$

- All measures the `density' of a graph

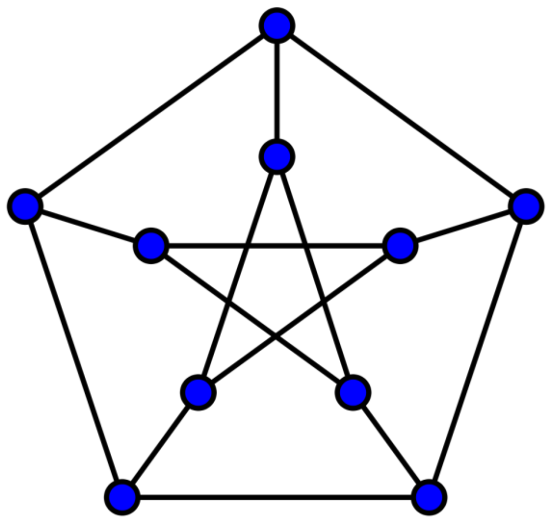
- $d(G) \geq \delta(G)$

Distance and diameter

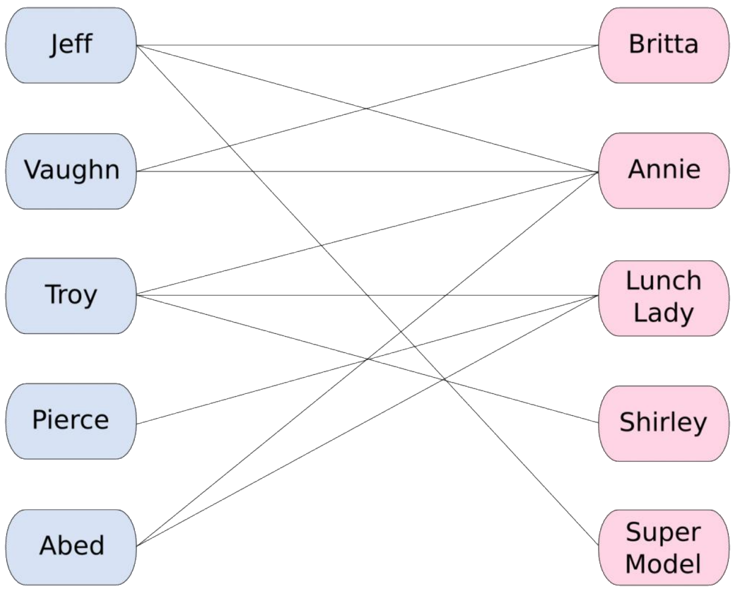
- The distance $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x{\sim}y$ path
  - if no such path exists, we set $d(x, y) \coloneqq \infty$
- The greatest distance between any two vertices in G is the diameter of G

Girth

- The minimum length of a cycle in a graph G is the girth $g(G)$ of G
- Example: The Peterson graph is the unique 5-cage
  - cubic graph (every vertex has degree 3)
  - girth = 5
  - smallest graph satisfies the above properties
- A tree has girth ∞



Bipartite graphs



- Theorem (1.2.18, W, Kőnig 1936)
  A graph is bipartite $\Longleftrightarrow$ it contains no odd cycle

Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Trees

Definition and properties

- A tree is a connected graph T with no cycles
- Recall that a graph is bipartite $\Leftrightarrow$ it has no odd cycle
- (Ex 3, S1.3.1, H) A tree of order $n \geq 2$ is a bipartite graph

- Recall that an edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
- $\Rightarrow$ Every edge in a tree is a bridge
- T is a tree $\Leftrightarrow$ T is minimally connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$

Equivalent definitions (Theorem 1.5.1, D)

- T is a tree of order n
  $\Leftrightarrow$ Any two vertices of T are linked by a unique path in T
  $\Leftrightarrow$ T is minimally connected
  - i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$

  $\Leftrightarrow$ T is maximally acyclic
  - i.e. T contains no cycle but $T + xy$ does for any non-adjacent vertices $x, y \in T$

  $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n - 1$ edges

  $\Leftrightarrow$ (Theorem 1.13, H) T is acyclic with $n - 1$ edges

Leaves of tree

- A vertex of degree 1 in a tree is called a leaf
- Theorem (1.14, H; Ex9, S1.3.2, H) Let T be a tree of order $n \geq 2$. Then T has at least two leaves
- (Ex3, S1.3.2, H) Let T be a tree with max degree Δ. Then T has at least Δ leaves
- (Ex10, S1.3.2, H) Let T be a tree of order $n \geq 2$. Then the number of leaves is

$$2 + \sum_{v:d(v)\geq 3} (d(v) - 2)$$

- (Ex8, S1.3.2, H) Every nonleaf in a tree is a cut vertex

Properties

- The center of a tree
- Theorem (1.15, H) In any tree, the center is either a single vertex or a pair of adjacent vertices
- Tree as subgraphs
- Theorem (1.16, H) Let T be a tree of order $k + 1$ with k edges. Let G be a graph with $\delta(G) \geq k$. Then G contains T as a subgraph

Spanning tree

- Given a graph G and a subgraph T, T is a spanning tree of G if T is a tree that contains every vertex of G
- Example: A telecommunications company tries to lay cable in a new neighbourhood
- Proposition (2.1.5c, W) Every connected graph contains a spanning tree

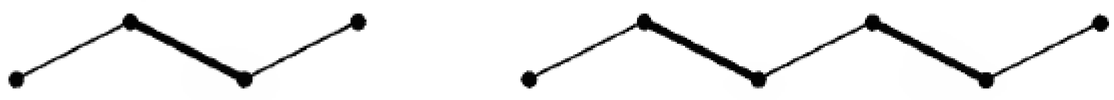
Matchings

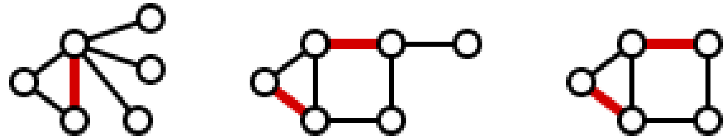
Definitions

- A matching is a set of independent edges, in which no pair shares a vertex
- The vertices incident to the edges of a matching M are M-saturated; the others are M-unsaturated
- A perfect matching in a graph is a matching that saturates every vertex
- Example (3.1.2, W) The number of perfect matchings in $K_{n,n}$ is $n!$
- Example (3.1.3, W) The number of perfect matchings in K_{2n} is
$$f_n = (2n-1)(2n-3)\cdots 1 = (2n-1)!!$$

Maximal/maximum matchings 极大/最大

- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge
- A maximum matching is a matching of maximum size among all matchings in the graph
- Example: P_3, P_5





- Every maximum matching is maximal, but not every maximal matching is a maximum matching

Stable matching

- A family $(\leq_v)_{v \in V}$ of linear orderings $\leq_v$ on $E(v)$ is a set of preferences for G
- A matching M in G is stable if for any edge $e \in E \setminus M$, there exists an edge $f \in M$ such that e and f have a common vertex v with $e <_v f$
  - Unstable: There exists $xy \in E \setminus M$ but $xy', x'y \in M$ with $xy' <_x xy$ $x'y <_y xy$

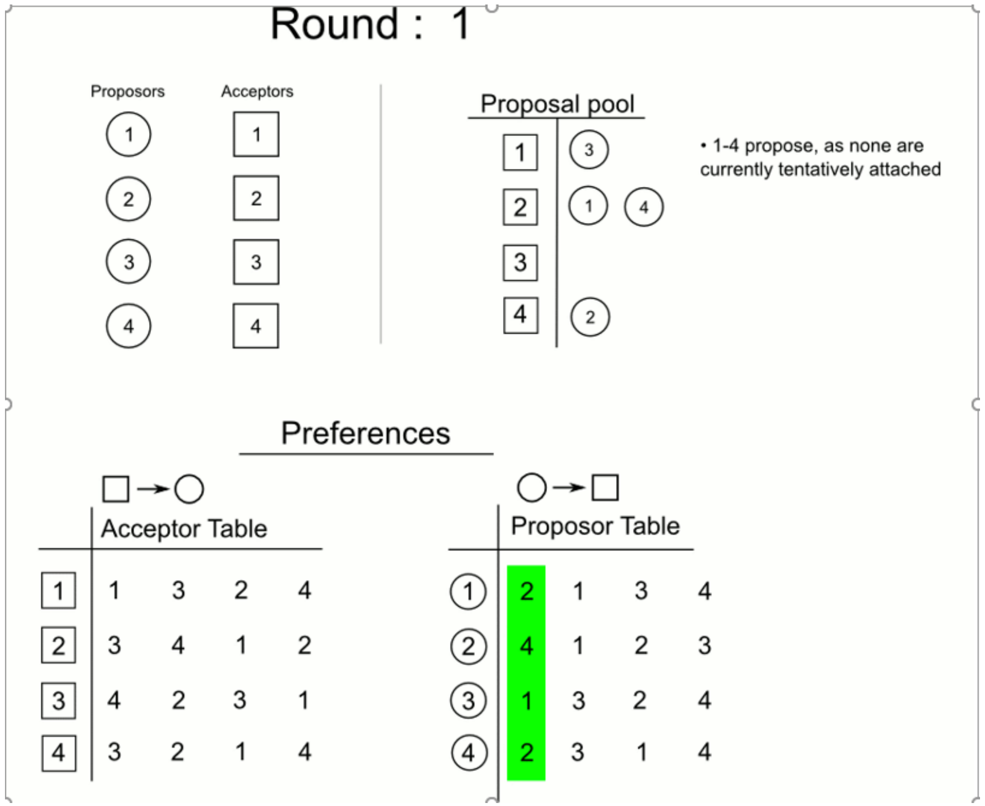
3.2.16. Example. Given men x, y, z, w, women a, b, c, d, and preferences listed below, the matching $\{xa, yb, zd, wc\}$ is a stable matching. ■

Men $\{x, y, z, w\}$	Women $\{a, b, c, d\}$
$x : a > b > c > d$	$a : z > x > y > w$
$y : a > c > b > d$	$b : y > w > x > z$
$z : c > d > a > b$	$c : w > x > y > z$
$w : c > b > a > d$	$d : x > y > z > w$

Gale-Shapley Proposal Algorithm

- **Input**: Preference rankings by each of n men and n women
- **Idea**: Produce a stable matching using proposals by maintaining information about who has proposed to whom and who has rejected whom
- **Iteration**: Each man proposes to the highest woman on his preference list who has not previously rejected him
  - If each woman receives exactly one proposal, stop and use the resulting matching
  - Otherwise, every woman receiving more than one proposal rejects all of them except the one that is highest on her preference list
  - Every woman receiving a proposal says “maybe” to the most attractive proposal received

Example


Round : 1
Proposors
Acceptors
Proposal pool
• 1-4 propose, as none are currently tentatively attached
Preferences
Acceptor Table
1 1 3 2 4
2 3 4 1 2
3 4 2 3 1
4 3 2 1 4
Proposor Table
1 2 1 3 4
2 4 1 2 3
3 1 3 2 4
4 2 3 1 4

Theoretical guarantee for the Proposal Algorithm

- Theorem (3.2.18, W, Gale-Shapley 1962) The Proposal Algorithm produces a stable matching
- Who proposes matters (jobs/candidates)
- When the algorithm runs with women proposing, every woman is as least as happy as when men do the proposing
  - And every man is at least as unhappy

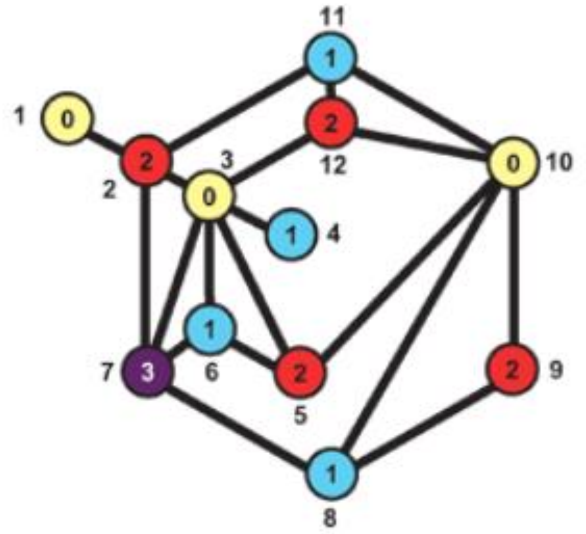
3.2.16. Example. Given men x, y, z, w, women a, b, c, d, and preferences listed below, the matching $\{xa, yb, zd, wc\}$ is a stable matching. ■

Men $\{x, y, z, w\}$	Women $\{a, b, c, d\}$
$x : a > b > c > d$	$a : z > x > y > w$
$y : a > c > b > d$	$b : y > w > x > z$
$z : c > d > a > b$	$c : w > x > y > z$
$w : c > b > a > d$	$d : x > y > z > w$

Coloring

Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k, a k-coloring is a function $K: V(G) \longrightarrow \{1, \ldots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k “colors,” then K is an assignment of one color to each vertex.
- We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G, we say that G is k-colorable

Chromatic number

- Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable
- Examples

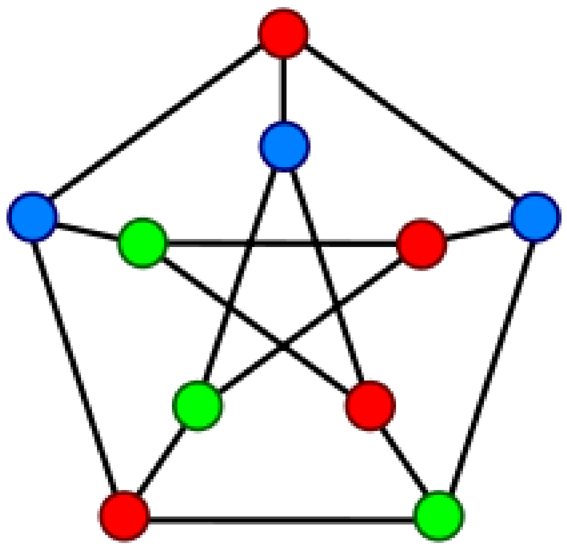
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$

$$\chi(E_n) = 1,$$

$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite $\Leftrightarrow$ it is 2-colorable

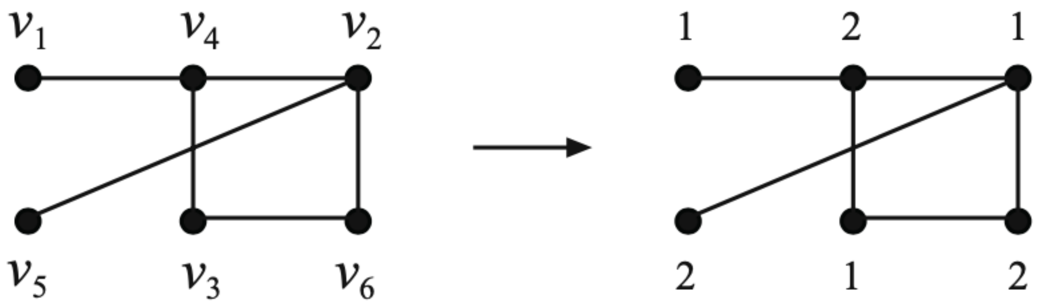
Bounds on Chromatic number

- Theorem (1.41, H) For any graph G of order n, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \ldots, v_n$
- Next, order the available colors $(1,2, \ldots, n)$ in some way
  - Start coloring by assigning color 1 to vertex v_1
  - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1
  - To color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors

Examples: Different orders result in different number of colors

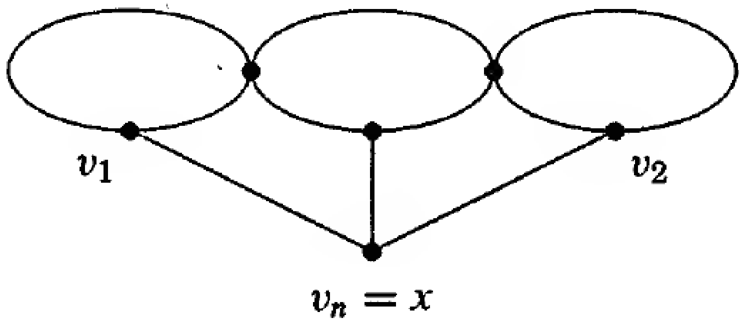


Bound of the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
- The equality is obtained for complete graphs and cycles with an odd number of vertices

Brooks's theorem

- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$



Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$

- Theorem (1.44, H) For any graph G, $\chi(G) \geq \omega(G)$

Chromatic number and independence number

- Theorem (1.45, H; Ex6, S1.6.2, H) For any graph G of order n,
$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The Four Color Problem

- Q: Is it true that the countries on any given map can be colored with four or fewer colors in such a way that adjacent countries are colored differently?
- Theorem (Four Color Theorem) Every planar graph is 4-colorable
- Theorem (Five Color Theorem) (1.47, H) Every planar graph is 5-colorable

Theorem 1.35. *If G is a planar graph, then G contains a vertex of degree at most five. That is, $\delta(G) \leq 5$.*

Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $c_G(k)$ to be the number of different colorings of a graph G using at most k colors
- Examples:
  - How many different colorings of K_4 using 4 colors?
    - $4 \times 3 \times 2 \times 1$
    - $c_{K_4}(4) = 24$
  - How many different colorings of K_4 using 6 colors?
    - $6 \times 5 \times 4 \times 3$
    - $c_{K_4}(6) = 360$
  - How many different colorings of K_4 using 2 colors?
    - 0
    - $c_{K_4}(2) = 0$

Examples

- If $k \geq n$

$$c_{K_n}(k) = k(k-1)\cdots(k-n+1)$$

- If $k < n$

$$c_{K_n}(k) = 0$$

- G is k-colorable $\Leftrightarrow\ \chi(G) \leq k\ \Leftrightarrow c_G(k) > 0$
- $\chi(G) = \min\{k \geq 1 : c_G(k) > 0\}$

Chromatic recurrence

- $G - e$ and G/e

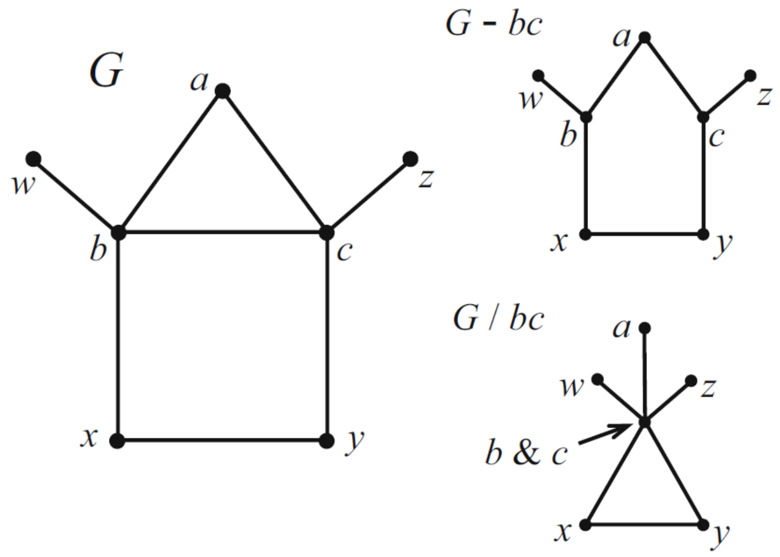


FIGURE 1.98. Examples of the operations.

- Theorem (1.48, H; 5.3.6, W) Let G be a graph and e be any edge of G. Then

$$c_G(k) = c_{G-e}(k) - c_{G/e}(k)$$

Use chromatic recurrence to compute $c_G(k)$

- Example: Compute $c_{P_3}(k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $c_{P_3}(1) = 0, c_{P_3}(2) = 2$

FIGURE 1.102. Two 2-colorings of P_3

More examples

- Path P_{n-1} has $n-1$ edges (n vertices)
$$c_{P_{n-1}}(k) = k(k-1)^{n-1}$$
- Any tree T on n vertices
$$c_T(k) = k(k-1)^{n-1}$$
- Cycle C_n
$$c_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$
  - When n is odd, $c_{C_n}(2) = 0, c_{C_n}(3) > 0$
  - When n is even, $c_{C_n}(2) > 0$

Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
  - $c_G(k)$ is a polynomial in k of degree n
  - The leading coefficient of $c_G(k)$ is 1
  - The constant term of $c_G(k)$ is 0
    - If G has i components, then the coefficients of $k^0, \dots, k^{i-1}$ are 0
    - G is connected $\Leftrightarrow$ the coefficient of k is nonzero
  - The coefficients of $c_G(k)$ alternate in sign
  - The coefficient of the k^{n-1} term is $-|E(G)|$
    - A graph G is a tree $\Leftrightarrow c_G(k) = k(k-1)^{n-1}$

      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n-1$ edges

    - A graph G is complete $\Leftrightarrow c_G(k) = k(k-1)\cdots(k-n+1)$

Planarity

Definition and examples

- A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices
- If G has no such representation, G is called nonplanar
- A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G

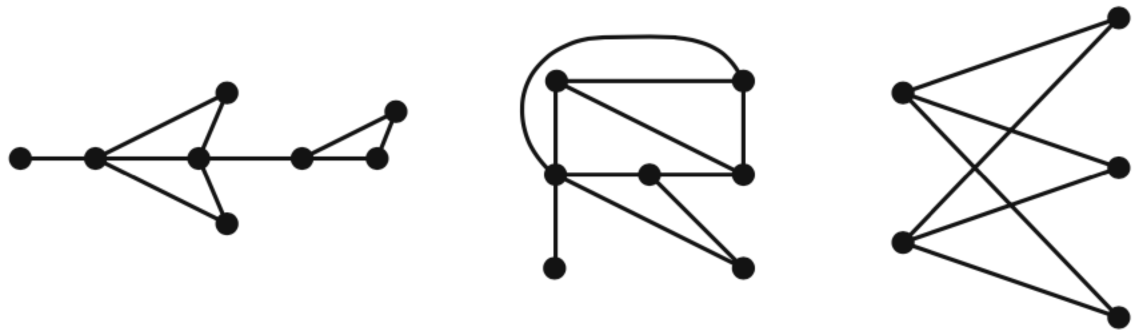
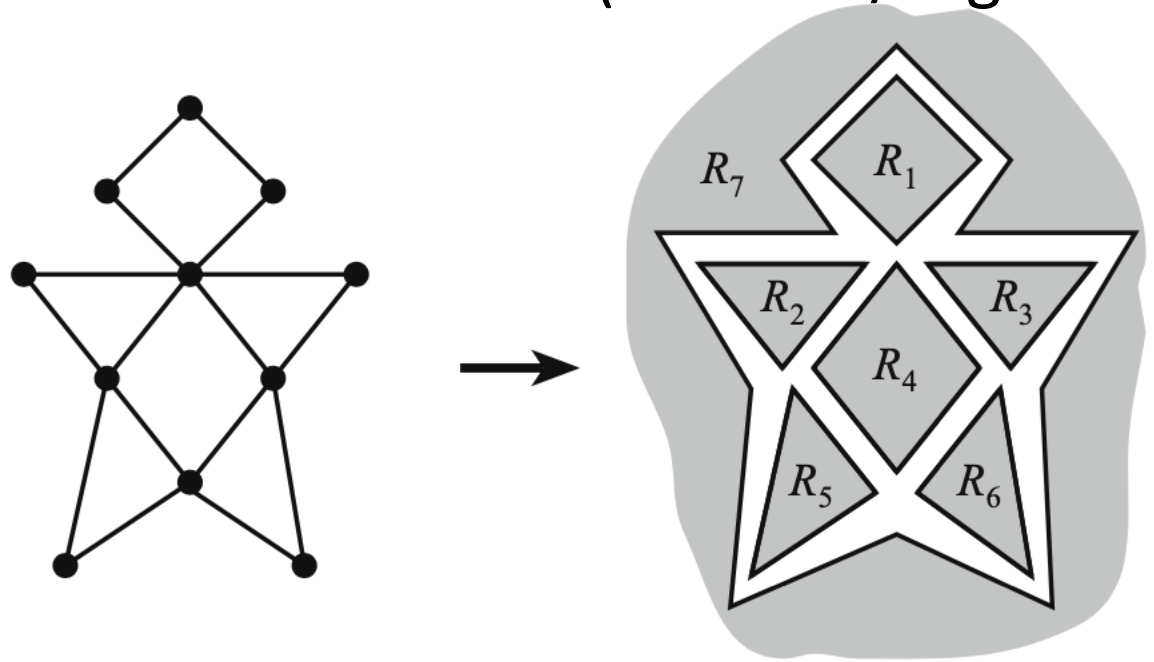


FIGURE 1.73. Examples of planar graphs.

Region

- Given a planar representation of a graph G, a region is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G
- The region R_7 is called the exterior (or outer) region



An edge bounds a region

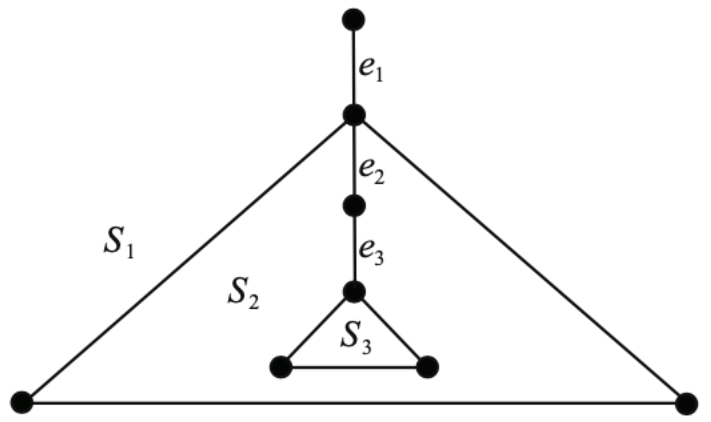
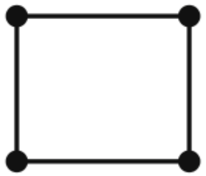


FIGURE 1.76. Edges e_1, e_2, and e_3 touch one region only.

- An edge can come into contact with either one or two regions
- Example:
  - Edge e_1 is only in contact with one region S_1
  - Edge e_2, e_3 are only in contact with S_2
  - Each of other edges is in contact with two regions
- An edge e bounds a region R if e comes into contact with R and with a region different from R
- The bounded degree $b(R)$ is the number of edges that bound the region
  - Example: $b(S_1) = b(S_3) = 3, b(S_2) = 6$

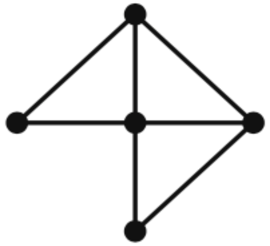
The relationship between numbers of vertices, edges and regions

- The number of vertices n
- The number of edges m
- The number of regions r

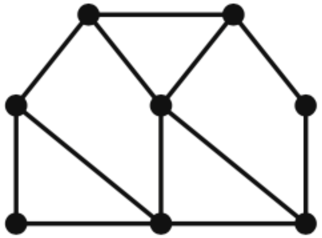


$n = 4$
$m = 4$
$r = 2$

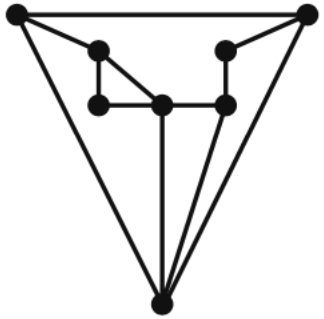
$n = 7$
$m = 9$
$r = 4$



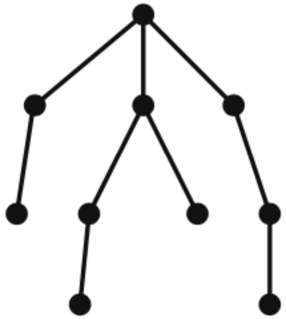
$n = 5$
$m = 7$
$r = 4$



$n = 8$
$m = 12$
$r = 6$



$n = 8$
$m = 12$
$r = 6$



$n = 10$
$m = 9$
$r = 1$

Euler's formula

- Theorem (1.31, H; Euler 1748) If G is a connected planar graph with n vertices, m edges, and r regions, then
$$n - m + r = 2$$
  - Need Lemma: (Ex4, S1.5.1, H) Every tree is planar
- (Ex6, S1.5.2, H) Let G be a planar graph with k components. Then
$$n - m + r = k + 1$$

$K_{3,3}$ is nonplanar

- Theorem (1.32, H) $K_{3,3}$ is nonplanar

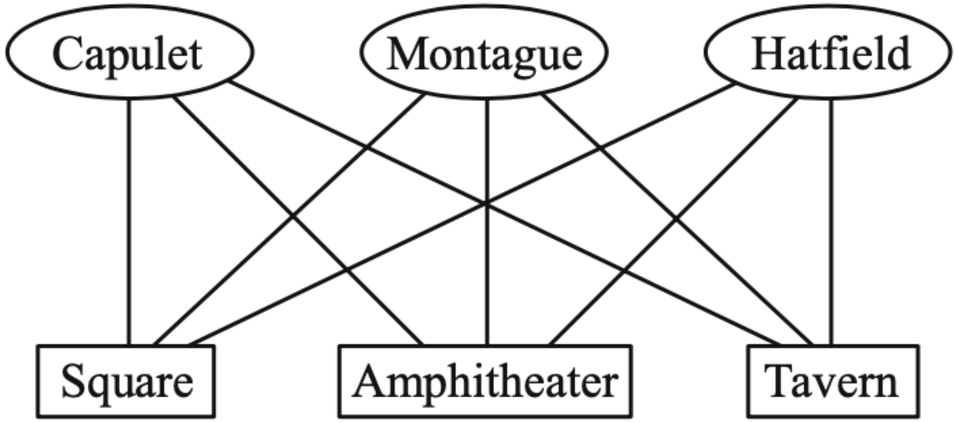


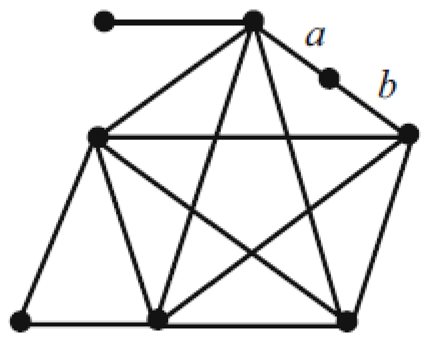
FIGURE 1.72. Original routes.

Upper bound for m

- Theorem (1.33, H) If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$. Furthermore, if equality holds, then every region is bounded by 3 edges.
- (Ex4, S1.5.2, H) Let G be a connected, planar, K_3-free graph of order $n \geq 3$. Then G has no more than $2n - 4$ edges
- Corollary (1.34, H) K_5 is nonplanar
- Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$
- (Ex5, S1.5.2, H) If G is bipartite planar graph, then $\delta(G) < 4$

Subdivision 细分

- A subdivision of an edge e in G is a substitution of a path for e
- A graph H is a subdivision of G if H can be obtained from G by a finite sequence of subdivisions
- Example:
  - The graph on the right contains a subdivision of K_5
  - In the below, H is a subdivision of G



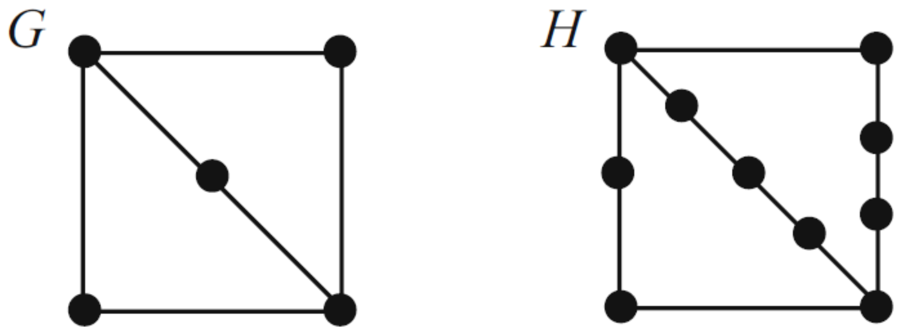


FIGURE 1.84. A graph and a subdivision.

Kuratowski's Theorem

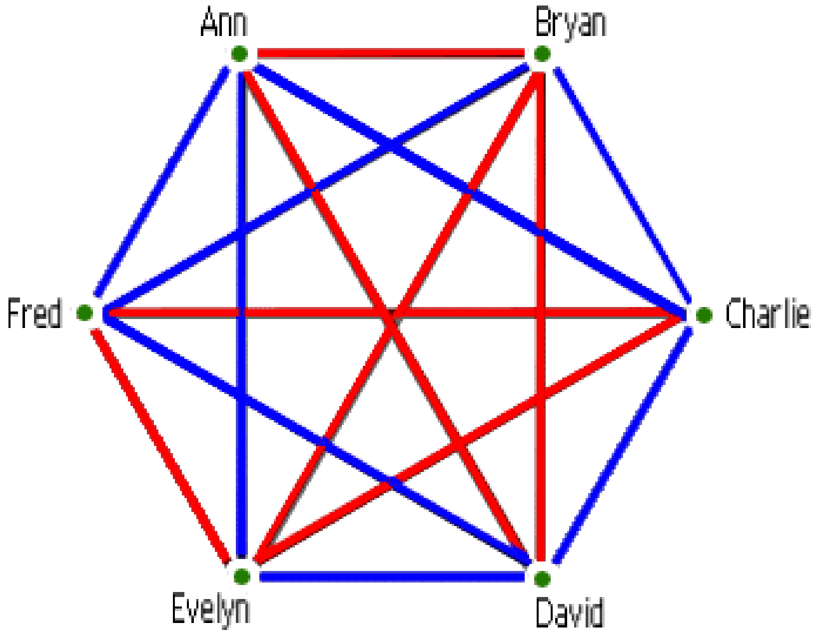
- Theorem (1.39, H; Ex1, S1.5.4, H) A graph G is planar $\Leftrightarrow$ every subdivision of G is planar

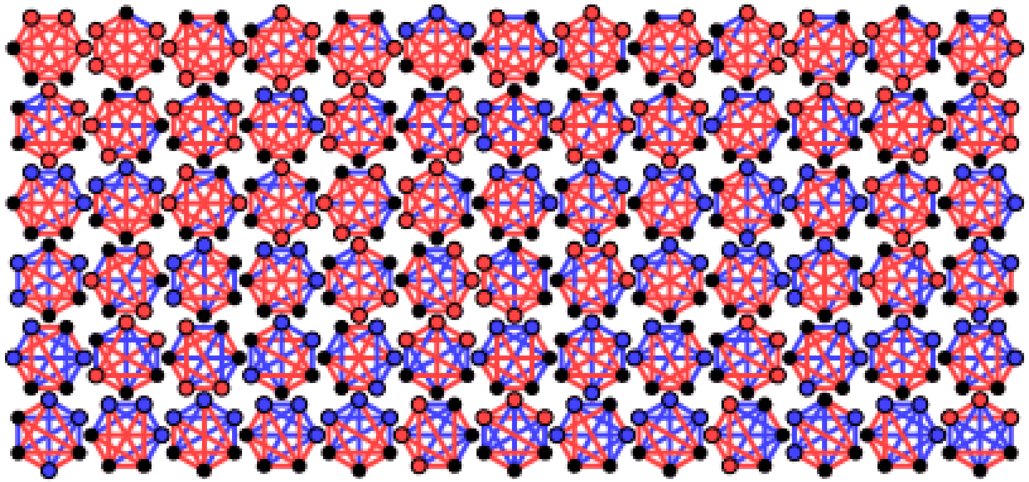
- Theorem (1.40, H; Kuratowski 1930) A graph is planar $\Leftrightarrow$ it contains no subdivision of $K_{3,3}$ or K_5

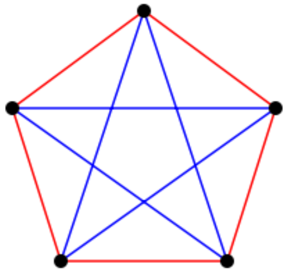
Ramsey Theory

The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?





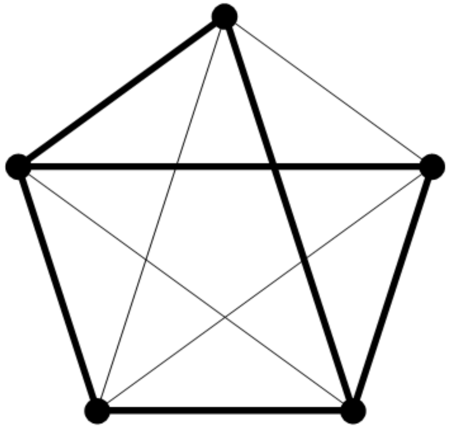


R(3,3)=6
R(3,4)=R(4,3)=9
R(3,5)=R(5,3)=14
R(3,6)=R(6,3)=18

https://plus.maths.org/content/friends-and-strangers
Wikipedia

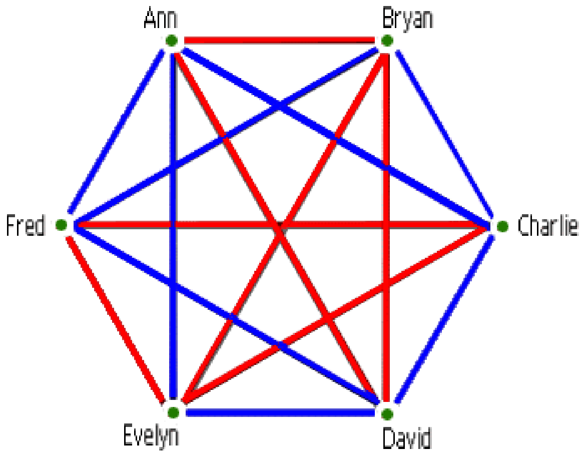
(classical) Ramsey number

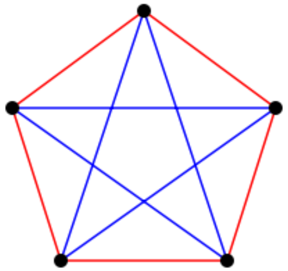
- A 2-coloring of the edges of a graph G is any assignment of one of two colors of each of the edges of G
- Let p and q be positive integers. The (classical) Ramsey number associated with these integers, denoted by $R(p, q)$, is defined to be the smallest integer n such that every 2-coloring of the edges of K_n either contains a red K_p or a blue K_q as a subgraph
- It is a typical problem of extremal graph theory



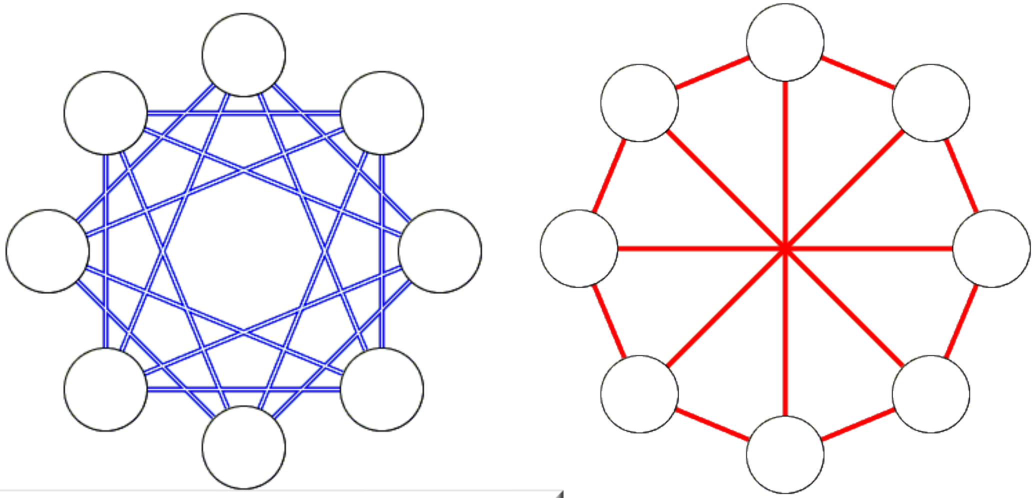
Examples

- $R(1,3) = 1$
- (Ex2, S1.8.1, H) $R(1,k) = 1$
- $R(2,4) = 4$
- (Ex3, S1.8.1, H) $R(2,k) = k$
- Theorem (1.61, H) $R(3,3) = 6$





Examples (cont.)



- Theorem (1.62, H) $R(3,4) = 9$

> **Theorem** A finite graph G has an even number of vertices with odd degree

- (Ex4, S1.8.1, H) $R(p,q) = R(q,p)$

Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[10]	36–41	49–61	59[14]–84	73–115	92–149
5					**43–48**	58–87	80–143	101–216	133–316	149[14]–442
6						**102–165**	115[14]–298	134[14]–495	183–780	204–1171
7							**205–540**	217–1031	252–1713	292–2826
8								**282–1870**	329–3583	343–6090
9									**565–6588**	581–12677
10										**798–23556**

Bounds on Ramsey numbers

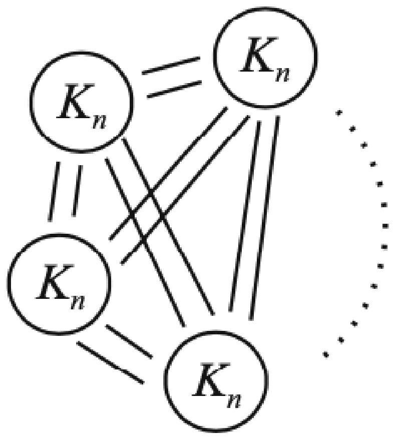
- Theorem (1.64, H; 2.28, H) If $q \geq 2, q \geq 2$, then
$$R(p,q) \leq R(p-1,q) + R(p,q-1)$$
Furthermore, if both terms on the RHS are even, then the inequality is strict

  Theorem A finite graph G has an even number of vertices with odd degree

- Theorem (1.63, H; 2.29, H) $R(p,q) \leq \binom{p+q-2}{p-1}$
- Theorem (1.65, H) For integer $q \geq 3$, $R(3,q) \leq \frac{q^2+3}{2}$
- Theorem (1.66, H; Erdős and Szekeres 1935)
If $p \geq 3$, $R(p,p) > \lfloor 2^{p/2} \rfloor$

Graph Ramsey Theory

- Given two graphs G and H, define the graph Ramsey number $R(G, H)$ to be the smallest value of n such that any 2-coloring of the edges of K_n contains either a red copy of G or a blue copy of H
  - The classical Ramsey number $R(p, q)$ would in this context be written as $R(K_p, K_q)$
- Theorem (1.67, H) If G is a graph of order p and H is a graph of order q, then $R(G, H) \leq R(p, q)$
- Theorem (1.68, H) Suppose the order of the largest component of H is denoted as $C(H)$.
  Then $R(G, H) \geq (\chi(G) - 1)(C(H) - 1) + 1$



Graph Ramsey Theory (cont.)

- Theorem (1.69, H) $R(T_m, K_n) = (m-1)(n-1)+1$

Theorem (1.45, H; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

A graph G is called *k-critical* if $\chi(G) = k$ and $\chi(G - v) < k$ for each vertex v of G.

(a) Find all 1-critical and 2-critical graphs.

(b) Give an example of a 3-critical graph.

(c) If G is k-critical, then show that G is connected.

(d) If G is k-critical, then show that $\delta(G) \geq k - 1$.

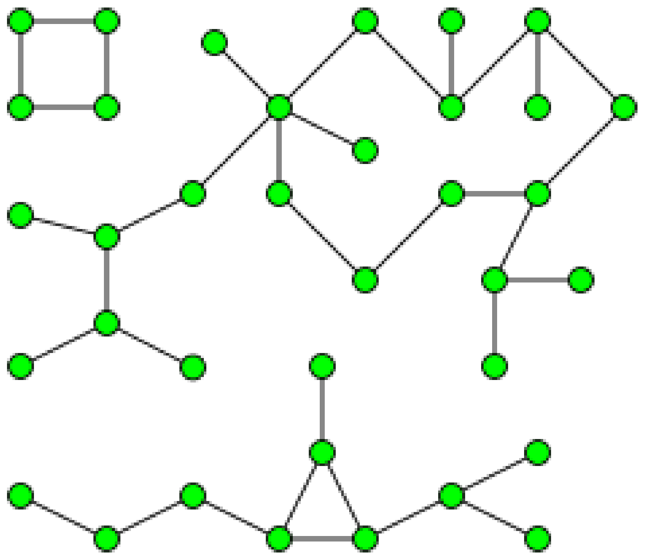
(e) Find all of the 3-critical graphs. Hint: Use part (d).

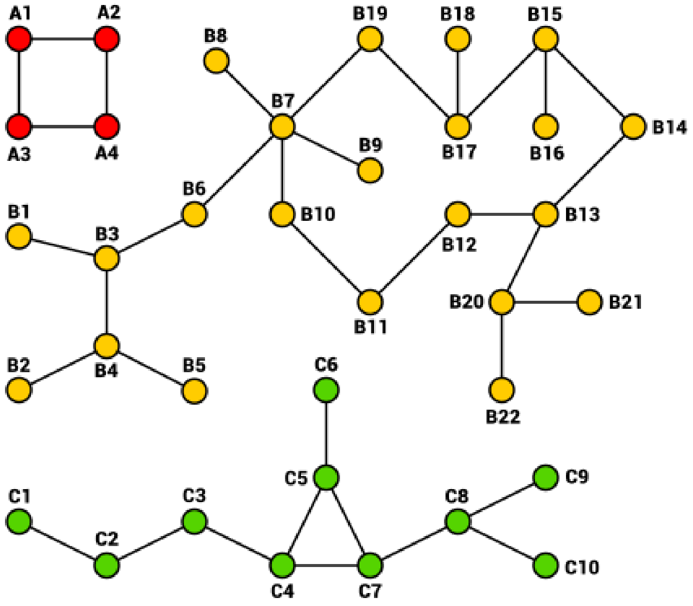
Theorem (1.16, H) Let T be a tree of order $k+1$ with k edges. Let G be a graph with $\delta(G) \geq k$. Then G contains T as a subgraph

Connectivity

Connected, connected component

- A graph G is connected if $G \neq \emptyset$ and any two of its vertices are linked by a path
- A maximal connected subgraph of G is a (connected) component



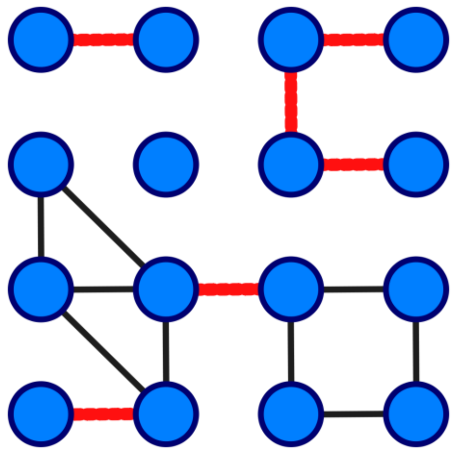


Connected vs. minimal degree

- Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then G is connected
- (Ex16, S1.1.2, H) (1.3.16, W)
  If $\delta(G) \geq \frac{n-2}{2}$, then G need not be connected

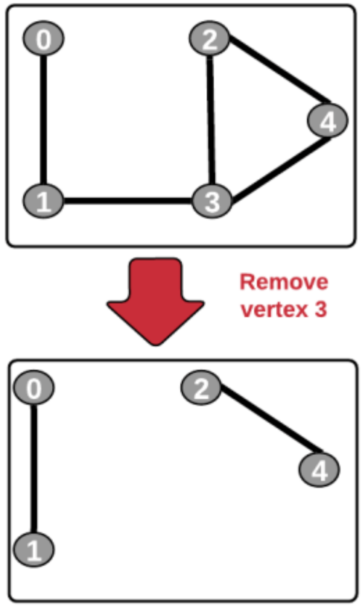
- Extremal problems
- “best possible” “sharp”

Add/delete an edge



- Components are pairwise disjoint; no two share a vertex
- Adding an edge decreases the number of components by 0 or 1
  - $\Rightarrow$ deleting an edge increases the number of components by 0 or 1
- Proposition (1.2.11, W)
  Every graph with n vertices and k edges has at least $n - k$ components
- An edge e is called a bridge if the graph $G - e$ has more components
- Proposition (1.2.14, W)
  An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
  - Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G

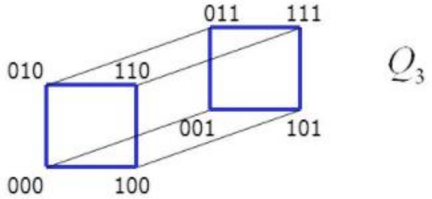
Cut vertex and connectivity



- A node v is a cut vertex if the graph $G - v$ has more components
- A proper subset S of vertices is a vertex cut set if the graph $G - S$ is disconnected
- The connectivity, $\kappa(G)$, is the minimum size of a cut set of G
  - The graph is k-connected for any $k \leq \kappa(G)$

Connectivity properties

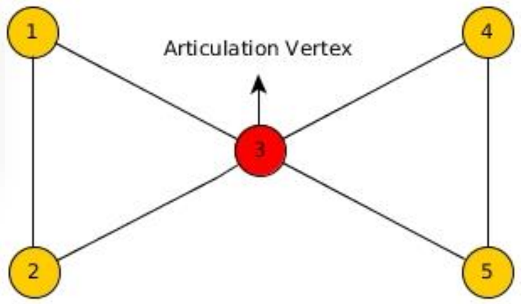
- $\kappa(K^n) := n - 1$
- If G is disconnected, $\kappa(G) = 0$
  - $\Rightarrow$ A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If G is connected, non-complete graph of order n, then
$$1 \leq \kappa(G) \leq n - 2$$
- For convention, $\kappa(K_1) = 0$
- Example (4.1.3, W) For k-dimensional cube $Q_k = \{0,1\}^k$, $\kappa(Q_k) = k$



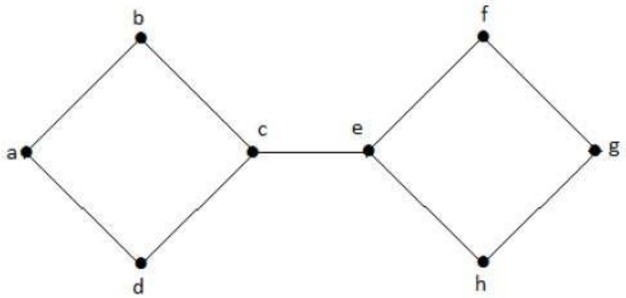
Connectivity properties (cont.)

Proposition (1.2.14, W)
An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G

- Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G

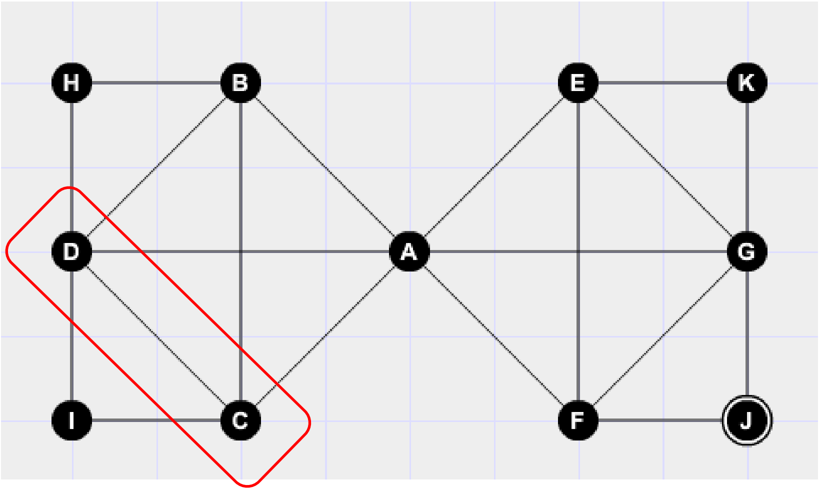


- $\kappa(G) \geq 2 \Leftrightarrow G$ is connected and has no cut vertices
- A vertex lies on a cycle $\not\Leftrightarrow$ it is not a cut vertex
  - $\Rightarrow$ (Ex13, S1.1.2, H) Every vertex of a connected graph G lies on at least one cycle $\not\Rightarrow \kappa(G) \geq 2$
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle
- (Ex12, S1.1.2, H) G has a cut vertex vs. G has a bridge

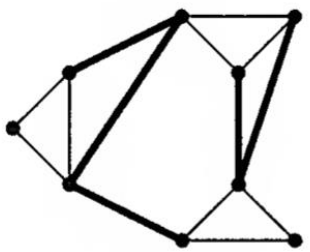


Connectivity and minimal degree

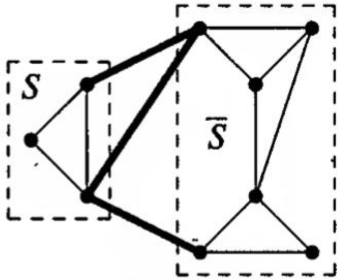
- (Ex15, S1.1.2, H)
- $\kappa(G) \leq \delta(G)$
- If $\delta(G) \geq n-2$, then $\kappa(G) = \delta(G)$



Edge-connectivity



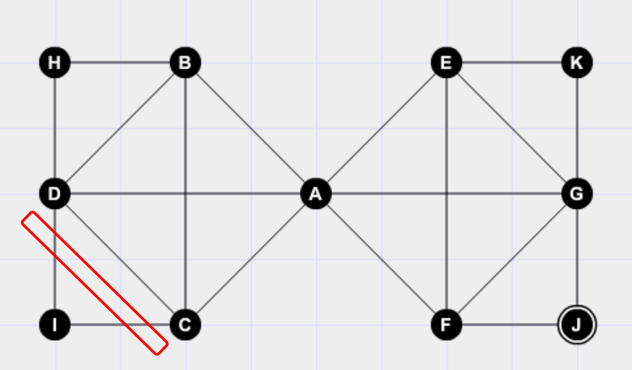
disconnecting set



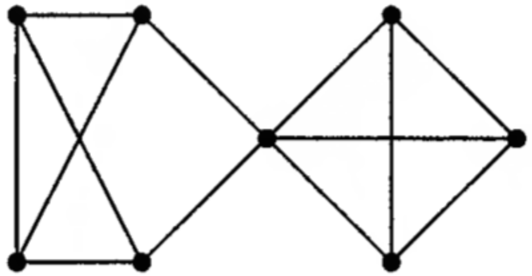
edge cut

- A disconnecting set of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component
  - A graph is k-edge-connected if every disconnecting set has at least k edges
  - The edge-connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set
  - $\lambda(G) = 0$ if G is disconnected
- Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one endpoint in S and the other in T
  - An edge cut is an edge set of the form $[S, S^c]$ where S is a nonempty proper subset of $V(G)$
- Every edge cut is a disconnecting set, but not vice versa
- Every minimal disconnecting set of edges is an edge cut

Connectivity and edge-connectivity



- Proposition (1.4.2, D) If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$
- Example (4.1.10, W) Possibility of $\kappa(G) < \lambda(G) < \delta(G)$



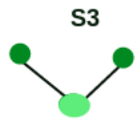
- Theorem (4.1.11, W) If G is a 3-regular graph, then $\kappa(G) = \lambda(G)$

Properties of edge cut

- When $\lambda(G) < \delta(G)$, a minimum edge cut cannot isolate a vertex
- Similarly for edge cut
- Proposition (4.1.12, W) If S is a set of vertices in a graph G, then
$$|[S, S^c]| = \sum_{v \in S} d(v) - 2e(G[S])$$
- Corollary (4.1.13, W) If G is a simple graph and $|[S, S^c]| < \delta(G)$ for some nonempty proper subset S of $V(G)$, then $|S| > \delta(G)$

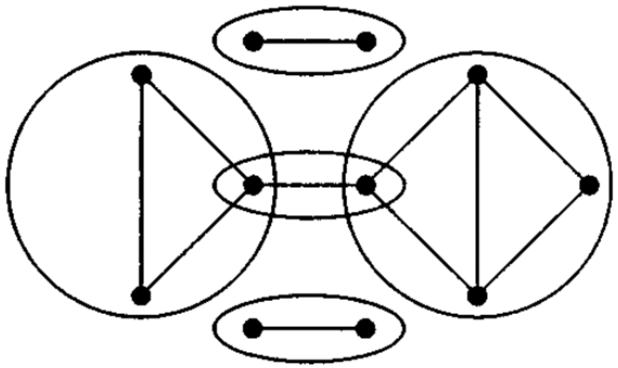
Bond

- An edge cut may contain another edge cut
- Example: $K_{1,2}$ or star graphs
- A bond is a minimal nonempty edge cut
- Proposition (4.1.15, W) If G is a connected graph, then an edge cut F is a bond $\Leftrightarrow$ $G - F$ has exactly two components



Blocks

- A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block
- Example



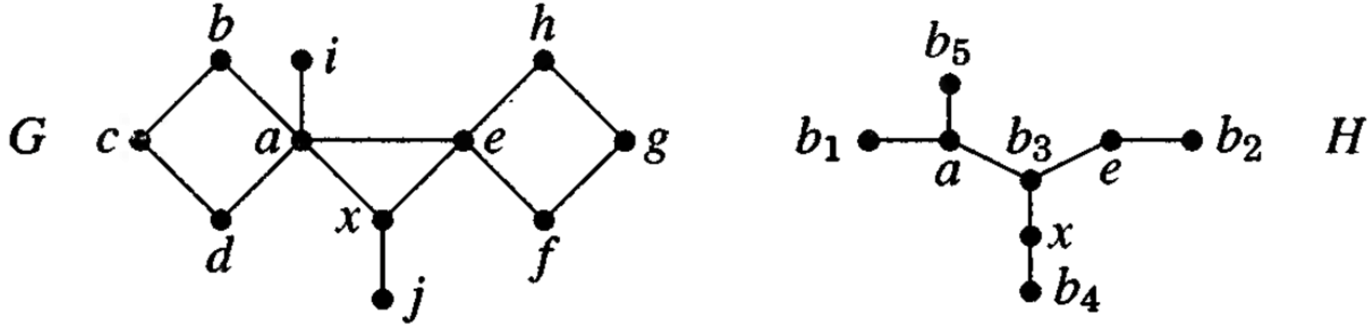
- An edge of a cycle cannot itself be a block
  - An edge is block $\Longleftrightarrow$ it is a bridge
  - The blocks of a tree are its edges
- If a block has more than two vertices, then it is 2-connected
  - The blocks of a loopless graph are its isolated vertices, bridges, and its maximal 2-connected subgraphs

Intersection of two blocks

- Proposition (4.1.19, W) Two blocks in a graph share at most one vertex
  - When two blocks share a vertex, it must be a cut-vertex
- Every edge is a subgraph with no cut-vertex and hence is in a block. Thus blocks in a graph decompose the edge set

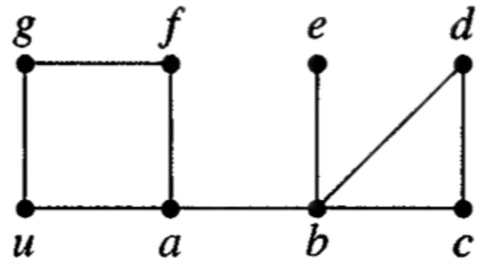
Block-cutpoint graph

- The block-cutpoint graph of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of $H \Leftrightarrow v \in B_i$



- (Ex34, S4.1, W) When G is connected, its block-cutpoint graph is a tree

Depth-first search (DFS)



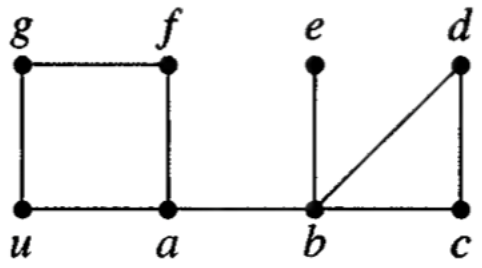
- Depth-first search

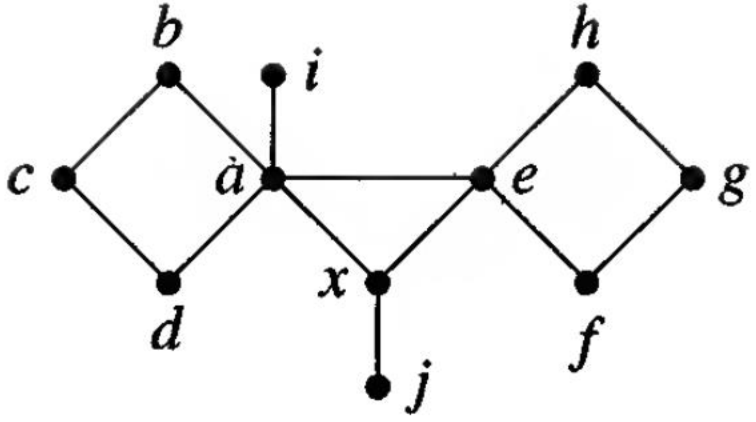
- Lemma (4.1.22, W) If T is a spanning tree of a connected graph grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T

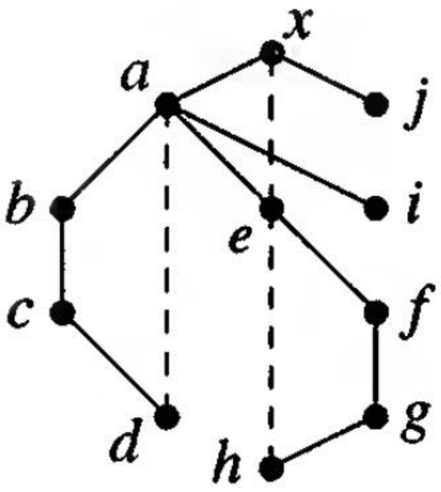
Finding blocks by DFS

- **Input**: A connected graph G
- **Idea**: Build a DFS tree T of G, discarding portions of T as blocks are identified. Maintain one vertex called ACTIVE
- **Initialization**: Pick a root $x \in V(H)$; make x ACTIVE; set $T = \{x\}$
- **Iteration**: Let v denote the current active vertex
  - If v has an unexplored incident edge vw, then
    - If $w \notin V(T)$, then add vw to T, mark vw explored, make w ACTIVE
    - If $w \in V(T)$, then w is an ancestor of v; mark vw explored
  - If v has no more unexplored incident edges, then
    - If $v \neq x$ and w is a parent of v, make w ACTIVE. If no vertex in the current subtree T' rooted at v has an explored edge to an ancestor above w, then $V(T') \cup \{w\}$ is the vertex set of a block; record this information and delete $V(T')$
    - if $v = x$, terminate

Example







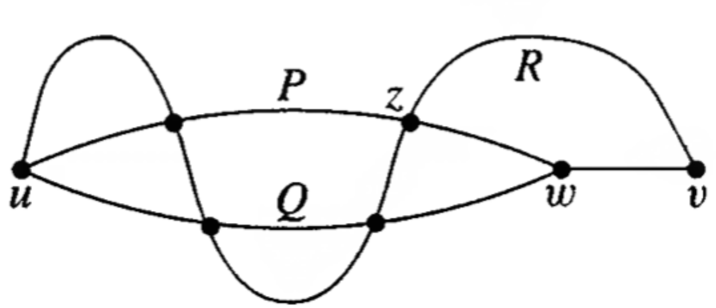
Strong orientation

- Theorem (2.5, L) Let G be a finite connected graph without bridges. Then G admits a strong orientation, i.e. an orientation that is a strongly connected digraph
  - A directed graph is strongly connected if for every pair of vertices (v, w), there is a directed path from v to w

- The blocks of a loopless graph are its isolated vertices, bridges, and its maximal 2-connected subgraphs

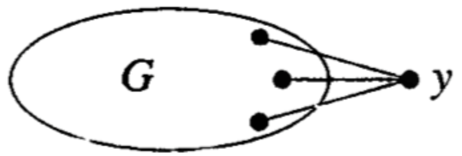
2-connected graphs

- Two paths from u to v are internally disjoint if they have no common internal vertex
- Theorem (4.2.2, W; Whitney 1932)
  A graph G having at least three vertices is 2-connected $\Leftrightarrow$ for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G

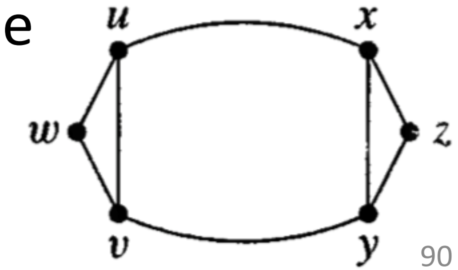


Equivalent definitions for 2-connected graphs

- Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



- Theorem (4.2.4, W) For a graph G with at least three vertices, TFAE
  - G is connected and has no cut-vertex
  - For all $x, y \in V(G)$, there are internally disjoint x, y-paths
  - For all $x, y \in V(G)$, there is a cycle through x and y
  - $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle



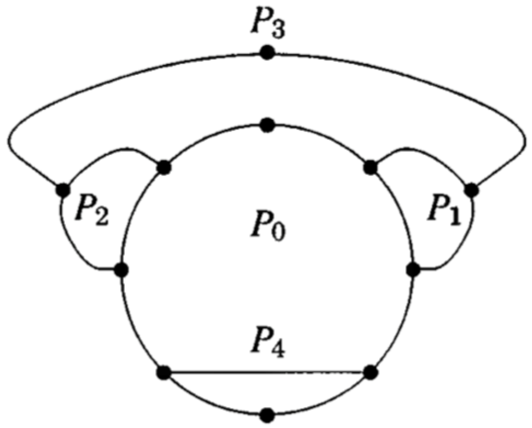
Subdivision keeps 2-connectivity

A subdivision of an edge e in G is a substitution of a path for e

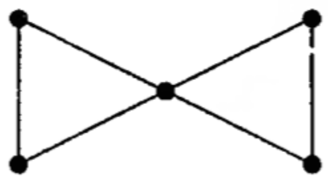
- Corollary (4.2.6, W) If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected

Ear decomposition

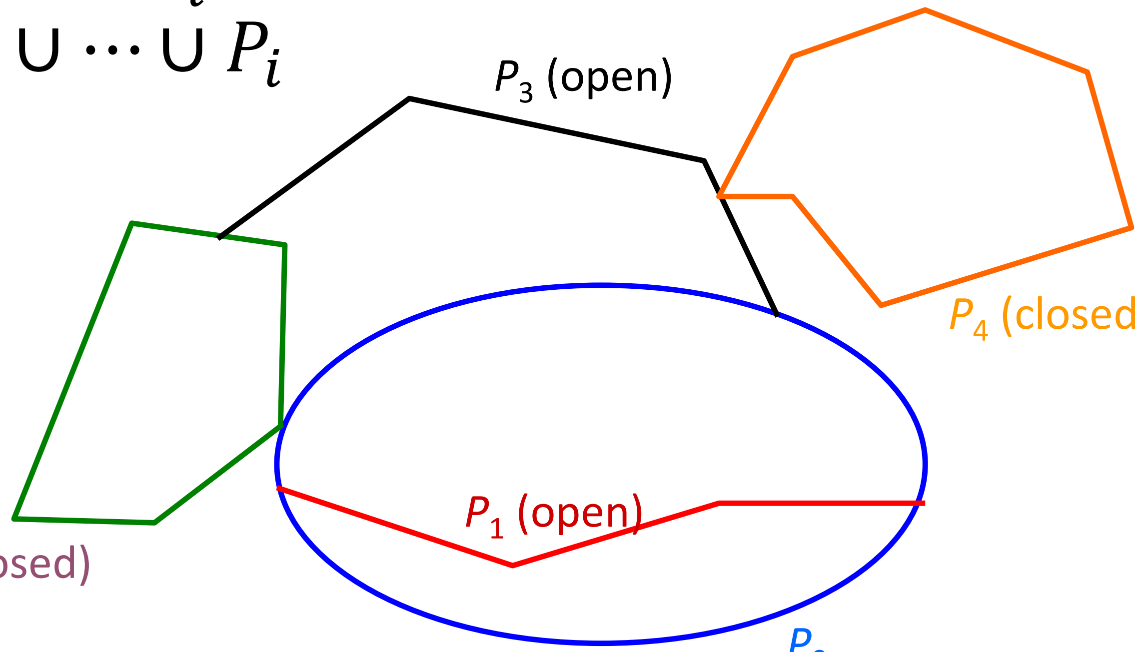
- An ear of a graph G is a maximal path whose internal vertices have degree 2 in G
- An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$
- Theorem (4.2.8, W)
  A graph is 2-connected $\Leftrightarrow$ it has an ear decomposition.
  Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition



Closed-ear



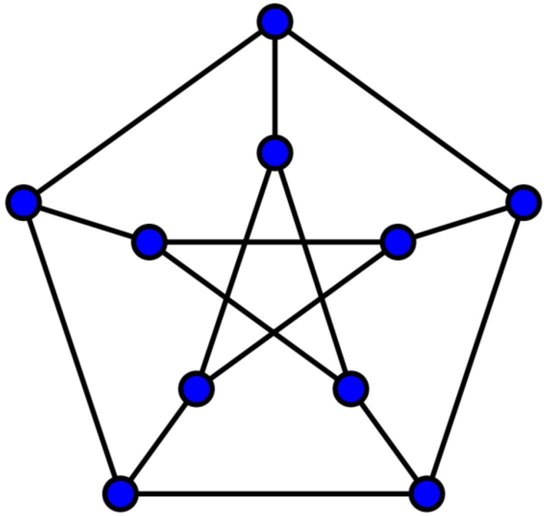
- A closed ear of a graph G is a cycle C such that all vertices of C except one have degree 2 in G
- A closed-ear decomposition of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an (open) ear or a closed ear in $P_0 \cup \dots \cup P_i$



Closed-ear decomposition

- Theorem (4.2.10, W)
  A graph is 2-edge-connected $\Leftrightarrow$ it has a closed-ear decomposition.
  Every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition

Peterson graph



- The Peterson graph is the unique 5-cage
  - cubic graph (every vertex has degree 3)
  - girth = 5
  - smallest graph satisfies the above properties
- $\kappa = 3, \alpha = 4$
- Radius=2, diameter=2
- Has a Hamiltonian path but no Hamiltonian cycle
- Chromatic number is 3
- Connectivity is 3, edge-connectivity is 3